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On The System of Double Diophantine Equations

$$x+y=u^2, \frac{x}{D}+y=v^2$$

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Abstract :

The thrust of this paper is to obtain many non-zero distinct integer values of x such that $x+y=u^2, \frac{x}{D}+y=v^2$ where $D\geq 0$ & square-free and y is a known integer. A few numerical examples are given. The recurrence relation satisfied by the values of x is presented.

Keywords : System of double Diophantine equations ,Diophantine problem

Introduction:

The theory of Diophantine equation is a treasure house in which the search for many hidden relations and properties form a treasure hunt. In fact, Diophantine problems dominated most of the celebrated unsolved mathematical problems. Certain diophantine problems come from physical problems or from immediate mathematical generalizations and others come from geometry in a variety of ways .Certain diophantine problems are neither trivial nor difficult to analyze. In this context ,one may refer [1-19].The above results motivated us to search for the integer solutions to some other choices of double diophantine equations. In this paper , many non-zero distinct integer values of x such that $x+y=u^2, \frac{x}{D}+y=v^2$ where $D\geq 0$ & square-free and y is a known integer ,are obtained. A few numerical examples are given. The recurrence relation satisfied by the values of x is presented.

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Method of analysis:

The system of equations to be solved is

$$x + y = u^2 \tag{1}$$

$$\frac{x}{D} + y = v^2 \tag{2}$$

where D is a non-zero integer >1 and square-free

The elimination of x between (1) and (2) gives

$$u^2 = Dv^2 - y(D-1) \tag{3}$$

Let $(u_\alpha^0, v_\alpha^0, y_\alpha)$ be an integral solution of (3) for given D and the corresponding value of x is

$$x_0 = (u_\alpha^0)^2 - y_\alpha$$

To obtain the other values of x, we proceed as follows:

Let $(\tilde{u}_n, \tilde{v}_n)$ be the general solution of the Pellian

$$u^2 = Dv^2 + 1 \tag{4}$$

given by

$$\tilde{u}_n + \sqrt{D}\,\tilde{v}_n = (\tilde{u}_0 + \sqrt{D}\,\tilde{v}_0)^{n+1}, \ n = 0,1,2,....$$

in which $(\tilde{u}_0, \tilde{v}_0)$ is the initial solution of (4).

Applying the lemma of Brahmagupta between the solutions (u_α^0, v_α^0) and $(\tilde{u}_n, \tilde{v}_n)$, the sequence of values of u and v satisfying (3) are given by

$$v_n = u_\alpha^0 \tilde{v}_n + v_\alpha^0 \tilde{u}_n$$

$$v_n = \frac{1}{2\sqrt{D}}\left\{ (u_\alpha^0 + \sqrt{D}v_\alpha^0)(\tilde{u}_0 + \sqrt{D}\,\tilde{v}_0)^{n+1} - (u_\alpha^0 - \sqrt{D}v_\alpha^0)(\tilde{u}_0 - \sqrt{D}\,\tilde{v}_0)^{n+1} \right\}$$

$$u_n = u_\alpha^0 \tilde{u}_n + D v_\alpha^0 \tilde{v}_n$$

$$u_n = \frac{1}{2}\left\{ (u_\alpha^0 + \sqrt{D}v_\alpha^0)(\tilde{u}_0 + \sqrt{D}\,\tilde{v}_0)^{n+1} + (u_\alpha^0 - \sqrt{D}v_\alpha^0)(\tilde{u}_0 - \sqrt{D}\,\tilde{v}_0)^{n+1} \right\}$$

Substituting the values of u_n and y_α in (1), the sequence of values of x are obtained as

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$$x_{n+1} = u_n^2 - y_\alpha \ , \ n = 0,1,2,....$$

The values of x satisfy the following recurrence relation

$$\sqrt{x_{n+2} + y_\alpha} - 2\tilde{u}_0 \sqrt{x_{n+1} + y_\alpha} + \sqrt{x_n + y_\alpha} = 0$$

To analyze the nature of the solutions one has to go in for particular values of D and y.

A few illustrations are given below:

ILLUSTRATION I:

Let $D = 3$.Then

(I) : $y_\alpha = \alpha^2 + 2\alpha - 2 \ , \ u_\alpha^0 = \alpha + 4 \ , \ v_\alpha^0 = \alpha + 2 \ , \ \tilde{u}_0 = 2 \ , \ \tilde{v}_0 = 1$

Thus the values of x satisfying (1) and (2) are given by

$$x_0 = 6\alpha + 18$$

$$x_{n+1} = \frac{1}{4}\left\{\left((\alpha+4)+(\alpha+2)\sqrt{3}\right)\left(2+\sqrt{3}\right)^{n+1} + \left((\alpha+4)-(\alpha+2)\sqrt{3}\right)\left(2-\sqrt{3}\right)^{n+1}\right\}^2 - \left(\alpha^2 + 2\alpha - 2\right)$$

Some numerical examples are:

Table: 1

n	Values of x when		
	$\alpha = 1$ $y_1 = 1$	$\alpha = 2$ $y_2 = 6$	$\alpha = 3$ $y_3 = 13$
-1	24	30	36
0	360	570	828
1	5040	8094	11868

(II) : $y_\alpha = \alpha^2 + 2\alpha + 1 \ , \ u_\alpha^0 = \alpha + 1 \ , \ v_\alpha^0 = \alpha + 1 \ , \ \tilde{u}_0 = 2 \ , \ \tilde{v}_0 = 1$

The values of x satisfying (1) and (2) are given by

$$x_0 = 0$$

$$x_{n+1} = \frac{1}{4}\left\{\left((\alpha+1)+(\alpha+1)\sqrt{3}\right)\left(2+\sqrt{3}\right)^{n+1} + \left((\alpha+1)-(\alpha+1)\sqrt{3}\right)\left(2-\sqrt{3}\right)^{n+1}\right\}^2 - \left(\alpha^2 + 2\alpha + 1\right)$$

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A few values of x are presented below:

Table: 2

n	Values of x when		
	$\alpha=1$ $y_1=4$	$\alpha=2$ $y_2=9$	$\alpha=3$ $y_3=16$
0	96	216	384
1	1440	3240	5760
2	20160	45360	80640
3	280896	632016	1123584
4	39124080	8803080	15649920

The above solutions in (I) and (II) satisfying the following recurrence relation

$$\sqrt{x_{n+2}+y_{\alpha}}-4\sqrt{x_{n+1}+y_{\alpha}}+\sqrt{x_{n}+y_{\alpha}}=0$$

ILLUSTRATION 2:

Let $D=5$.

(III) : $y_{\alpha}=\alpha^2$, $u_{\alpha}^{0}=\alpha$, $v_{\alpha}^{0}=\alpha$, $\tilde{u}_0=9$, $\tilde{v}_0=4$

A few values of x are presented below:

Table: 3

n	**Values of x when**		
	$\alpha=1$ $y_1=1$	$\alpha=2$ $y_2=4$	$\alpha=3$ $y_3=9$
0	**840**	**3360**	**7560**
1	**271440**	**1085760**	**2442960**
2	**87403800**	**349615200**	**786634200**

(IV) : $y_\alpha = \alpha^2 + 5\alpha + 5$, $u_\alpha^0 = \alpha$, $v_\alpha^0 = \alpha + 2$, $\tilde{u}_0 = 9$, $\tilde{v}_0 = 4$

Table: 4

n	Values of x when		
	$\alpha = 1$ $y_1 = 11$	$\alpha = 2$ $y_2 = 19$	$\alpha = 3$ $y_3 = 29$
0	4750	9585	16100
1	1540070	3104625	5212060
2	495908350	999697905	1678295060

The above patterns in (III) and (IV) satisfy the recurrence relation

$$\sqrt{x_{n+2} + y_\alpha} - 18\sqrt{x_{n+1} + y_\alpha} + \sqrt{x_n + y_\alpha} = 0$$

ILLUSTRATION 3:

Let $D = 6$.Then

(V) : $y_\alpha = \alpha^2$, $u_\alpha^0 = \alpha$, $v_\alpha^0 = \alpha$, $\tilde{u}_0 = 5$, $\tilde{v}_0 = 2$

Table: 5

n	Values of x when		
	$\alpha = 1$ $y_1 = 1$	$\alpha = 2$ $y_2 = 4$	$\alpha = 3$ $y_3 = 9$
0	288	1152	2592
1	28560	114240	257040
2	2798928	11195712	25190352

(VI) : $y_\alpha = [\alpha^2 + 4\alpha - 2]$, $u_\alpha^0 = [4 - \alpha]$,
$v_\alpha^0 = \alpha + 1$, $\tilde{u}_0 = 5$, $\tilde{v}_0 = 2$

Table: 6

n	Values of x when		
	$\alpha=1$ $y_1=3$	$\alpha=2$ $y_2=10$	$\alpha=3$ $y_3=19$
0	1518	2106	2790
1	149766	209754	279822
2	14676558	20557146	27426150

Patterns in (V) and (VI) satisfy the following recurrence relation:

$$\sqrt{x_{n+2}+y_{\alpha}}-10\sqrt{x_{n+1}+y_{\alpha}}+\sqrt{x_{n}+y_{\alpha}}=0$$

Other patterns of solutions for the system of equations (1) and (2) are derived

Below:

The elimination of y between (1) and (2) gives

$$u^2=v^2+\frac{(D-1)x}{D}$$

Substitution of $x=DX$ in the above equation leads to

$$u^2=v^2+(D-1)X$$

where D is either a square or a square-free integer , whose parametric solution is given by

$$u=\beta+D-1, v=\beta, X=2\beta+D-1$$

and hence

$$x=D(2\beta+D-1)$$

Using the values of x and u in (1) ,the corresponding value of y is given by

$$y=(\beta-1)^2-D \qquad (5)$$

A few examples are given below in Table: 7

Table :7 Examples

$\beta(=v)$	D	x	y	u
1	2	6	-2	2
2	3	18	-2	4
2	4	28	-3	5
3	5	50	-1	7
3	6	66	-2	8
5	7	112	9	11

Remarks :

Note that (5) is satisfied by

$$\beta = r^2 + s^2 + 1, D = 4r^2 s^2, y = (r^2 - s^2)^2$$

Thus, the pattern of solutions for the system of equations

$$x + y^2 = u^2, \frac{x}{D} + y^2 = v^2 \qquad (6)$$

is given by

$$x = 4r^2 s^2 (2r^2 + 2s^2 + 4r^2 s^2 + 1), y = r^2 - s^2, u = r^2 + s^2 + 4r^2 s^2, v = r^2 + s^2 + 1 \qquad (7)$$

where r, s are non-zero distinct parameters.

Again, introducing the transformations

$$r = p^2 + q^2, s = 2pq$$

in (7) , we obtain the solutions to the system of equations

$$x + y^4 = u^2, \frac{x}{D} + y^4 = v^2$$

Thus, the repetition of the above process yields the integer solutions to the system of equations in general and is given by

$$x + y^{2^n} = u^2, \frac{x}{D} + y^{2^n} = v^2$$

Albeit tacitly, the system of equations (6) is also satisfied by other sets of solutions which we present below :

The elimination of x in the system (6) results in

$$u^2 = Dv^2 - (D-1)y^2 \tag{8}$$

Introduction of the linear transformations

$$v = X \pm (D-1), y = X \pm D \tag{9}$$

in (8) leads to

$$X^2 - u^2 = D(D-1) \tag{10}$$

The above equation is solvable when $D = 4k, 4k+1$. For each value of D, there are two sets of integer solutions to the system of equations (6). For simplicity and brevity, the corresponding sets of integer solutions to (6) are presented below:

Solutions to (6) when $D = 4k$:

Set 1:

$$x = -4k(8k^2 + 6k + 1), y = (4k^2 + 3k + 1), u = (4k^2 - k - 1), v = (4k^2 + 3k)$$

Set 2:

$$x = (4k-2)(8k^2 - 6k), y = (4k^2 - 5k + 1), u = (4k^2 - k - 1), v = (4k^2 - 5k + 2)$$

Solutions to (6) when $D = 4k+1$:

Set 3:

$$x = -(32k^3 + 48k^2 + 22k + 3), y = (4k^2 + 5k + 2), u = (4k^2 + k - 1), v = (4k^2 + 5k + 1)$$

Set 4:

$$x = (32k^3 - 16k^2 - 2k + 1), y = (4k^2 - 3k), u = (4k^2 + k - 1), v = (4k^2 - 3k + 1)$$

Conclusion:

In this paper, an attempt has been made to obtain many non-zero distinct integer values of x such that $x + y = u^2, \frac{x}{D} + y = v^2$ where $D \geq 0$ & square-free and y is a known integer. One may search for other choices of system of double equations and determine their solutions

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